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1989 J. Phys. A: Math. Gen. 22 67

(http://iopscience.iop.org/0305-4470/22/1/013)

J. Phys. A: Math. Gen. **22** (1989) 67-70. Printed in the UK

Explicit derivation of the propagator for a Dirac delta potential

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Received 27 April 1988

Abstract. An explicit derivation of the propagator is given for a particle in a Dirac delta potential $\lambda\delta(x)$ by making a direct application of a variational technique followed by a systematic use of Laplace transforms. Although the solution of the corresponding Schrödinger equation has been known for years, the solution of the propagator has been obtained only recently.

1. Introduction

We give an explicit derivation of the propagator for a particle in a Dirac delta potential $\lambda\delta(x)$, where λ is a coupling parameter, by making a direct application of a variational technique followed by a systematic use of Laplace transforms. Although the solution of the corresponding Schrödinger equation has been known for years (cf Morse and Feshbach (1953); see also Lapidus (1987) for a recent treatment and variations), the solution of the propagator has been obtained only recently (Gaveau and Schulman (1986); see also Schulman (1986) for applications) by a procedure involving stochastic processes, Brownian motion and the Feynman-Kac formula (cf Kac 1959). The importance of the propagator approach to quantum mechanics was particularly emphasised by Feynman (1948) (see also Feynman and Hibbs 1965) and at present it is indisputably recognised as a very basic ingredient in considering dynamical problems in quantum mechanics.

For future reference we record here the expression for the free propagator ($\lambda = 0$) (Feynman 1948, Feynman and Hibbs 1965):

$$\langle xt|x't'\rangle_0 = \begin{cases} \sqrt{\dfrac{m}{2\pi i\hbar T}}\exp\left(-\dfrac{m(x-x')^2}{2i\hbar T}\right) & t > t' \\ 0 & t < t' \end{cases} \tag{1}$$

where $T = (t-t')$. For simplicity of the notation we will use units where $\hbar = 1$. The Laplace transform (cf Abramowitz and Stegun 1972) of $\langle xt|x't'\rangle_0$ is

$$\begin{aligned} L_0(x, x'; s) &= \int_0^\infty \mathrm{d}T\, \mathrm{e}^{-sT}\langle xt|x't'\rangle_0 \\ &= \sqrt{\frac{m}{2is}}\exp\left(-\sqrt{\frac{2ms}{i}}|x-x'|\right). \end{aligned} \tag{2}$$

This expression will be needed in the following.

2. Derivation of the propagator

The propagator $\langle xt|x't'\rangle$ satisfies a well known (cf Feynman 1948; Feynman and Hibbs 1965) partial differential equation:

$$\left(-\mathrm{i}\frac{\partial}{\partial t}-\frac{\Delta^2}{2m}+\lambda\delta(x)\right)\langle xt|x't'\rangle=-\mathrm{i}\delta(x-x')\delta(t-t') \tag{3}$$

and for its adjoint we have

$$\left(\mathrm{i}\frac{\partial}{\partial t}-\frac{\Delta^2}{2m}+\lambda\delta(x)\right)\langle x't'|xt\rangle=\mathrm{i}\delta(x-x')\delta(t-t'). \tag{4}$$

We take the partial derivation of (3) with respect to the coupling λ to obtain

$$\delta(x)\langle xt|x't'\rangle+\left(-\mathrm{i}\frac{\partial}{\partial t}-\frac{\Delta^2}{2m}+\lambda\delta(x)\right)\frac{\partial}{\partial\lambda}\langle xt|x't'\rangle=0. \tag{5}$$

Upon multiplying (5) by $\langle x''t''|xt\rangle$ from the left and integrating over x from $-\infty$ to ∞ ($-\infty<x<\infty$) and integrating over t from t' to t'' ($t'<t<t''$) we get

$$\int_{t'}^{t''}\mathrm{d}t\langle x''t''|0t\rangle\langle 0t|x't'\rangle$$
$$=-\int_{t'}^{t''}\mathrm{d}t\int_{-\infty}^{\infty}\mathrm{d}x\langle x''t''|\,xt\rangle\left(-\mathrm{i}\frac{\partial}{\partial t}-\frac{\Delta^2}{2m}+\lambda\delta(x)\right)\frac{\partial}{\partial\lambda}\langle xt|x't'\rangle. \tag{6}$$

We integrate the right-hand side of (6) by parts, and use (4) to obtain finally

$$\frac{\partial}{\partial\lambda}\langle x''t''|x't'\rangle=-\mathrm{i}\int_{t'}^{t''}\mathrm{d}t\langle x''t''|0t\rangle\langle 0t|x't'\rangle. \tag{7}$$

This is our basic starting equation. (Equation (7) is the content of the classic Schwinger dynamical principle (cf Schwinger 1951, 1961, Manoukian 1986), the knowledge of which, however, is not needed here.)

To solve (7) we set $\langle x''t''|x't'\rangle=F(x'',x';T)$, $T=t''-t'$, and define its Laplace transform:

$$L(x'',x';s)=\int_0^{\infty}\mathrm{d}T\,\mathrm{e}^{-sT}F(x'',x';T). \tag{8}$$

We consider first the case in (7) when $x''=0$, $x'=0$. That is, we consider first the equation

$$\frac{\partial}{\partial\lambda}F(0,0;T)=-\mathrm{i}\int_0^{T}\mathrm{d}t\,F(0,0;T-t)F(0,0;t). \tag{9}$$

This gives the Laplace-transformed equation

$$\frac{\partial}{\partial\lambda}L(0,0;s)=-\mathrm{i}L(0,0;s)^2 \tag{10}$$

whose solution is

$$L(0,0;s)=\frac{L_0(0,0;s)}{1+\mathrm{i}\lambda L_0(0,0;s)} \tag{11}$$

where $L_0(0, 0; s)$ is inferred from (2) to be equal to $\sqrt{m/2is}$. Hence the inverse Laplace transform $F(0, 0; T)$ is (cf Abramowitz and Stegun 1972):

$$F(0,0;\ T)=\sqrt{\frac{m}{2\mathrm{i}}}\left(\frac{1}{\sqrt{\pi T}}-b\exp(b^2T)\operatorname{erfc}(b\sqrt{T})\right) \tag{12}$$

$$b=\lambda\sqrt{\frac{m\mathrm{i}}{2}} \qquad \operatorname{erfc}(x)=\frac{2}{\sqrt{\pi}}\int_x^\infty \mathrm{d}u\,\exp(-u^2). \tag{13}$$

As the next step we consider the equation:

$$\frac{\partial}{\partial\lambda}\langle x''t''|0t'\rangle=-\mathrm{i}\int_{t'}^{t''}\mathrm{d}t\langle x''t''|0t\rangle\langle 0t|0t'\rangle \tag{14}$$

obtained from (7) by setting $x'=0$, where $\langle 0t|0t'\rangle$ is known and is given in (12) for $t-t'=T$. The Laplace-transformed equation of (14) is

$$\frac{\partial}{\partial\lambda}L(x'',0;\ s)=-\mathrm{i}L(x'',0;\ s)L(0,0,s) \tag{15}$$

whose solution is from (11)

$$L(x'',0;\ s)=L_0(x'',0;\ s)\left(1+\mathrm{i}\lambda\sqrt{\frac{m}{2\mathrm{i}s}}\right)^{-1} \tag{16}$$

and from (2), this becomes

$$L(x'',0;\ s)=\sqrt{\frac{m}{2\mathrm{i}s}}\exp\left(-\sqrt{\frac{2ms}{\mathrm{i}}}|x''|\right)\left(1+\mathrm{i}\lambda\sqrt{\frac{m}{2\mathrm{i}s}}\right)^{-1}. \tag{17}$$

Similarly we have

$$L(0,x'',s)=L(x'',0;\ s). \tag{18}$$

From (17) and (18) we may then readily obtain the solution of (7) for arbitrary x', x''. To this end, the Laplace-transformed equation of (7) is

$$\frac{\partial}{\partial\lambda}L(x'',x';\ s)=(-\mathrm{i})L(x'',0;\ s)L(0,x';\ s) \tag{19}$$

whose solution is

$$L(x'',x';\ s)=L_0(x'',x';\ s)-\frac{\lambda m}{2\sqrt{s}}\exp\left(-\sqrt{\frac{2m}{\mathrm{i}}}(|x'|+|x''|)\sqrt{s}\right)\left(\sqrt{s}+\lambda\sqrt{\frac{m\mathrm{i}}{2}}\right)^{-1}. \tag{20}$$

The inverse Laplace transform of (20) then gives (cf Abramowitz and Stegun 1972) the final result:

$$\langle x''t''|x't'\rangle=\langle x''t''|x't'\rangle_0-m\lambda\sqrt{\frac{m}{2\pi\mathrm{i}T}}\int_0^\infty\mathrm{d}u\,\exp(-um\lambda)\exp\left(-\frac{m}{2\mathrm{i}T}(|x''|+|x'|+u)^2\right) \tag{21}$$

where $T=t''-t'$, and $\langle x''t''|x't'\rangle_0$ is given in (1).

Acknowledgments

The author would like to thank Dr S D Jog for useful discussions. This work is supported by the Department of National Defence Award under CRAD no 3610-637:F4122.

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